PERMANENT CONFIGURATIONS IN THE PROBLEM OF FOUR BODIES*

BY

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1. **Introduction.** Two permanent configurations in the problem of three bodies have been known since the time of Lagrange, namely, the straight line and the equilateral triangle; each of these configurations exists whatever the masses may be. A complete generalization of the straight line configuration to n bodies was given by F. R. Moulton,† and some particular instances of other configurations have been given by R. Hoppe,‡ Andoyer,§ and W. R. Longley,|| each of which contains some element of symmetry.

In the present paper the problem of plane configurations (evidently, with the exception of the tetrahedron, there are no three-dimensional configurations) is removed from the field of differential equations to those of geometry and algebra by means of two theorems which hold for any number of bodies. The results of these theorems are used to give a complete and detailed analysis of the quadrilateral configurations of four bodies. It is evident that the method is applicable to any number of bodies.

2. **The differential equations.** Suppose there are n particles in the xy-plane which attract each other along the lines joining them according to any given function of the distance, but which, for simplicity, will be taken to be the inverse nth power of the distance. The differential equations, referred to the center of gravity of the system, are

$$x_i'' = -\sum m_j \frac{x_i - x_j}{r_{ij}^{n+1}} = -\sum m_j \frac{\cos\theta_{ij}}{r_{ij}^n},$$

$$y_i'' = -\sum m_j \frac{y_i - y_j}{r_{ij}^{n+1}} = -\sum m_j \frac{\sin\theta_{ij}}{r_{ij}^n} \quad (i = 1, \cdots, n; j \neq i);$$

or, in polar coördinates, in which

$$x_i = r_i \cos\theta_i, \qquad y_i = r_i \sin\theta_i,$$

* Presented to the Society, September 2, 1932; received by the editors May 16, 1932.

† *Periodic Orbits*, published by the Carnegie Institution of Washington, 1920, p. 285.

‡ *Erweiterung der bekannten Speciallösung des Dreikörperproblems*, Archiv der Mathematik und Physik, vol. 64, p. 218.

§ *Sur l'équilibre relatif de n corps*, Bulletin Astronomique, vol. 23 (1906), p. 50.

|| *Some particular solutions in the problem of n bodies*, Bulletin of the American Mathematical Society, vol. 13 (1906–07), p. 324.

the equations are

$$
(1)\qquad
\begin{aligned}
r_i'' - r_i\theta_i'^2 &= -\sum m_j \frac{\cos(\theta_{ij} - \theta_i)}{r_{ij}^n},\\
r_i\theta_i'' + 2r_i'\theta_i' &= -\sum m_j \frac{\sin(\theta_{ij} - \theta_i)}{r_{ij}^n}.
\end{aligned}
$$

In these equations θ_{ij} is the angle which the line r_{ij} makes with the x-axis.

If there exists a configuration which moves like a rigid system with the angular velocity ω, the mutual distances are all constants, say

$$r_i = l_i, \qquad r_{ij} = l_{ij};$$

and the angles are all linear functions of the time, say

$$\theta_i = \theta_i^{(0)} + \omega t, \qquad \theta_{ij} = \theta_{ij}^{(0)} + \omega t.$$

The differential equations reduce to

$$
(2)\qquad
\begin{aligned}
\sum m_j \frac{\cos(\theta_{ij}^{(0)} - \theta_i^{(0)})}{l_{ij}^n} &= \omega^2 l_i,\\
\sum m_j \frac{\sin(\theta_{ij}^{(0)} - \theta_i^{(0)})}{l_{ij}^n} &= 0 \qquad (i = 1, \cdots, n).
\end{aligned}
$$

The left members of these equations are the components of acceleration of the particle m_i, along the radius vector r_i and perpendicular to it, due to the attraction of all of the other bodies. Expressed in words these equations give the following theorem:

THEOREM I. *If a plane system of free particles, which is acted upon by no forces other than those of their mutual attractions, rotates about the center of gravity like a rigid system then the resultant acceleration of each of the particles, due to the attraction of all of the other particles, passes through the center of gravity of the system, and in magnitude is proportional to the distance of the particle from the center of gravity of the system; and, conversely, if there exists a plane configuration of n bodies in which the resultant acceleration of each particle passes through the center of gravity of the system and in magnitude is proportional to the distance of the particle from the center of gravity, then the system in this configuration can rotate like a rigid system.*

3. **The possibility of Keplerian motion.** The hypothesis of rigidity is stronger than is necessary, for the configuration is preserved if the system is altered in such a way that the ratios of the mutual distances are not changed; size and orientation being non-essentials.

Suppose one knows a configuration in which the resultant acceleration of each particle, due to the attraction of all of the others, is directed toward the center of gravity and in magnitude is proportional to the distance of the particle from the center of gravity, and equations (2), therefore, are satisfied.

Let ρ and θ be new variables, and in equations (1) take

$$r_i = \rho l_i, \qquad r_{ij} = \rho l_{ij},$$
$$\theta_i = \theta_i^{(0)} + \theta, \qquad \theta_{ij} = \theta_{ij}^{(0)} + \theta.$$

Equations (1) then become

$$l_i\rho'' - l_i\rho\theta'^2 = -\frac{1}{\rho^n}\sum m_j \frac{\cos(\theta_{ij}^{(0)} - \theta_i^{(0)})}{l_{ij}^n},$$
$$l_i(\rho\theta'' + 2\rho'\theta') = 0,$$

which, by virtue of equations (2) and removal of the factor l_i, become

$$\rho'' - \rho\theta'^2 = -\frac{\omega^2}{\rho^n},$$
$$\rho\theta'' + 2\rho'\theta' = 0.$$

These are the equations of motion of a particle which is attracted towards a fixed center by a force which varies inversely as the nth power of the distance. Hence

THEOREM II. *If a configuration of n particles exists for which motion as a rigid system is possible, then each particle of the system can move just as though it were attracted toward the center of gravity by a force which varies inversely as the nth power of the distance in such a way that the configuration is preserved.*

If the law of attraction is the Newtonian law and the configuration is such that circular motion is possible, then motion in a conic section in accordance with the laws of Kepler also is possible, the configuration being preserved throughout the motion.

PERMANENT QUADRILATERAL CONFIGURATIONS

4. **Vector equations.** Given four masses, m_1, m_2, m_3, m_4, and the quadrilateral of which they are the corners, Fig. 1. Let the line m_1m_2 be r_1, m_2m_3 be r_2, etc.; let the diagonal m_1m_3 be r_5 and the diagonal m_2m_4 be r_6. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ be unit vectors parallel to r_1, r_2, r_3, and r_4 respectively, taken as in the figure. The sides r_1 and r_3 intersect in the point A, and the sides r_2 and r_4 in the point B. Starting at m_i, let $\alpha_i r_i$ be the distance along the line r_i to the intersection with the opposite side (the point A or B). The ratio α_i is positive if this direction is the same as that of $\mathbf{a}_i$; otherwise, negative; and let $\beta_i = \alpha_i - 1$.

If L_{i+1} is the line which coincides with the side r_{i+1} of the quadrilateral, and if I_i is a variable vector with its origin at m_i, and t_i is a variable parameter, the vector equations of the lines L_{i+1} are

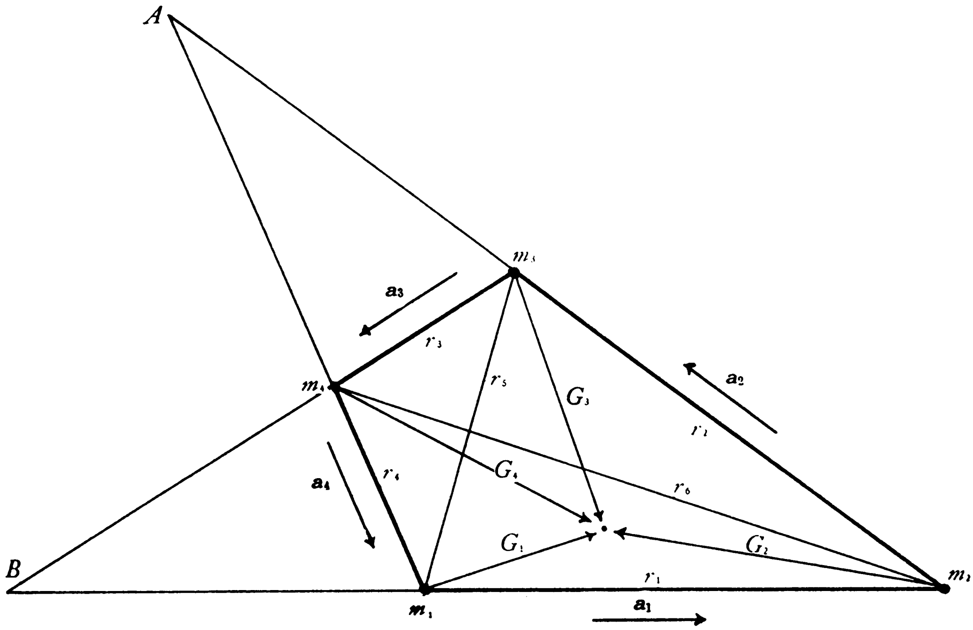

Fig. 1

$$(3)\qquad \begin{aligned} I_1 &= t_1 r_1 a_1 + (1 - t_1)\beta_4 r_4 a_4, \\ I_2 &= t_2 r_2 a_2 + (1 - t_2)\beta_1 r_1 a_1, \\ I_3 &= t_3 r_3 a_3 + (1 - t_3)\beta_2 r_2 a_2, \\ I_4 &= t_4 r_4 a_4 + (1 - t_4)\beta_3 r_3 a_3 \qquad (\beta_i = \alpha_i - 1, \alpha_i - \beta_i = +1). \end{aligned}$$

For certain values of the parameters t_i the vectors I_i coincide with the diagonals of the quadrilateral. These values are

$$t_i = \frac{\beta_{i+1}}{\alpha_{i+1}}, \quad 1 - t_i = \frac{1}{\alpha_{i+1}}, \text{ at the point } m_{i+2};$$

and therefore the vector expressions for the diagonals are

$$(4)\qquad \begin{aligned} I_{13} &= \frac{1}{\alpha_2}[\beta_2 r_1 a_1 + \beta_4 r_4 a_4], & I_{31} &= \frac{1}{\alpha_4}[\beta_4 r_3 a_3 + \beta_2 r_2 a_2], \\ I_{24} &= \frac{1}{\alpha_3}[\beta_3 r_2 a_2 + \beta_1 r_1 a_1], & I_{42} &= \frac{1}{\alpha_1}[\beta_1 r_4 a_4 + \beta_3 r_3 a_3]. \end{aligned}$$

Of course,

$$I_{13} + I_{31} = 0, \qquad I_{24} + I_{42} = 0.$$

Let M be the sum of the four masses, and let $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{G}_3$, and $\boldsymbol{G}_4$ be the vectors which represent the center of gravity of the system with respect to the masses m_1, m_2, m_3, and m_4 respectively. Then

$$\begin{aligned} M\boldsymbol{G}_1 &= m_2 r_1 \boldsymbol{a}_1 + m_3 \boldsymbol{I}_{13} - m_4 r_4 \boldsymbol{a}_4, \\ M\boldsymbol{G}_2 &= m_3 r_2 \boldsymbol{a}_2 + m_4 \boldsymbol{I}_{24} - m_1 r_1 \boldsymbol{a}_1, \\ M\boldsymbol{G}_3 &= m_4 r_3 \boldsymbol{a}_3 + m_1 \boldsymbol{I}_{31} - m_2 r_2 \boldsymbol{a}_2, \\ M\boldsymbol{G}_4 &= m_1 r_4 \boldsymbol{a}_4 + m_2 \boldsymbol{I}_{42} - m_3 r_3 \boldsymbol{a}_3; \end{aligned} \tag{5}$$

and the substitution of (4) in (5) gives

$$\begin{aligned} M\boldsymbol{G}_1 &= \frac{1}{\alpha_2}\big[(\alpha_2 m_2 + \beta_2 m_3) r_1 \boldsymbol{a}_1 + (\beta_4 m_3 - \alpha_2 m_4) r_4 \boldsymbol{a}_4\big], \\ M\boldsymbol{G}_2 &= \frac{1}{\alpha_3}\big[(\alpha_3 m_3 + \beta_3 m_4) r_2 \boldsymbol{a}_2 + (\beta_1 m_4 - \alpha_3 m_1) r_1 \boldsymbol{a}_1\big], \\ M\boldsymbol{G}_3 &= \frac{1}{\alpha_4}\big[(\alpha_4 m_4 + \beta_4 m_1) r_3 \boldsymbol{a}_3 + (\beta_2 m_1 - \alpha_4 m_2) r_2 \boldsymbol{a}_2\big], \\ M\boldsymbol{G}_4 &= \frac{1}{\alpha_1}\big[(\alpha_1 m_1 + \beta_1 m_2) r_4 \boldsymbol{a}_4 + (\beta_3 m_2 - \alpha_1 m_3) r_3 \boldsymbol{a}_3\big]. \end{aligned} \tag{6}$$

5. **Relations among the α's.** From the triangle $m_1 m_2 B$ it is found that

$$r_2 \boldsymbol{a}_2 = -\frac{1}{\alpha_2} r_1 \boldsymbol{a}_1 + \frac{\beta_4}{\alpha_2} r_4 \boldsymbol{a}_4,$$

and similarly

$$r_3 \boldsymbol{a}_3 = -\frac{1}{\alpha_3} r_2 \boldsymbol{a}_2 + \frac{\beta_1}{\alpha_3} r_1 \boldsymbol{a}_1.$$

These values substituted in the equation

$$r_1 \boldsymbol{a}_1 + r_2 \boldsymbol{a}_2 + r_3 \boldsymbol{a}_3 + r_4 \boldsymbol{a}_4 = 0$$

give the equation

$$\frac{1}{\alpha_2 \alpha_3}\big[(\beta_2 \beta_3 + \alpha_1 \alpha_2) r_1 \boldsymbol{a}_1 + (\beta_3 \beta_4 + \alpha_2 \alpha_3) r_4 \boldsymbol{a}_4\big] = 0.$$

Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_4$ are non-collinear vectors, it follows that, if $\alpha_2 \alpha_3 \neq \infty$,

$$\begin{aligned} \alpha_1 \alpha_2 + \beta_2 \beta_3 &= 0, \\ \alpha_2 \alpha_3 + \beta_3 \beta_4 &= 0; \end{aligned} \tag{7}$$

and, similarly,

$$\alpha_3\alpha_4 + \beta_4\beta_1 = 0,$$
$$\alpha_4\alpha_1 + \beta_1\beta_2 = 0.$$

Only two of these relations are independent, for if the first and fourth are solved for α_3 and α_4 and the results substituted in the second and third, both equations are satisfied identically.

From the differences of these equations one derives also

$$(7.5)\qquad \begin{aligned} & & \beta_1\alpha_2 + \beta_2\alpha_3 &= -1,\\ \alpha_2\beta_1 &= \alpha_4\beta_3, & \beta_2\alpha_3 + \beta_3\alpha_4 &= -1,\\ \alpha_3\beta_2 &= \alpha_1\beta_4, & \beta_3\alpha_4 + \beta_4\alpha_1 &= -1,\\ & & \beta_4\alpha_1 + \beta_1\alpha_2 &= -1. \end{aligned}$$

6. **Relations between sides and diagonals.** From the triangles $m_1m_2m_3$, Bm_1m_3, and Am_2m_3 are obtained

$$r_5^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(r_1r_2),$$
$$\alpha_3^2r_3^2 = \beta_1^2r_1^2 + r_2^2 + 2\beta_1r_1r_2\cos(r_1r_2),$$
$$\beta_4^2r_4^2 = r_1^2 + \alpha_2^2r_2^2 - 2\alpha_2r_1r_2\cos(r_1r_2).$$

The elimination of $\cos(r_1r_2)$ between the first and second, and between the first and third equations, gives two expressions for the diagonal r_5, namely

$$(8)\qquad \begin{aligned} \beta_1r_5^2 &= \alpha_1\beta_1r_1^2 + \alpha_1r_2^2 - \alpha_3^2r_3^2,\\ \alpha_2r_5^2 &= \beta_2r_1^2 - \alpha_2\beta_2r_2^2 + \beta_4^2r_4^2. \end{aligned}$$

Similarly, for the diagonal r_6,

$$(9)\qquad \begin{aligned} \beta_4r_6^2 &= \alpha_4r_1^2 + \alpha_4\beta_4r_4^2 - \alpha_2^2r_2^2,\\ \alpha_1r_6^2 &= -\alpha_1\beta_1r_1^2 + \beta_3^2r_3^2 + \beta_1r_4^2. \end{aligned}$$

The elimination of r_5^2 between the two equations of (8), or r_6^2 between the two equations of (9), leads, after some reduction, to the equation

$$(10)\qquad \begin{aligned} \beta_2(\beta_1r_1^2 + \alpha_2r_2^2) - \beta_4(\beta_3r_3^2 + \alpha_4r_4^2) &= 0, \quad \text{or}\\ \alpha_1(\beta_1r_1^2 + \alpha_2r_2^2) - \alpha_3(\beta_3r_3^2 + \alpha_4r_4^2) &= 0, \end{aligned}$$

since the determinant $\beta_2\alpha_3 - \alpha_1\beta_4$ is zero.

Since α_3 and α_4 are expressible in terms of α_1 and α_2 by means of (7), equation (10) is a relation between the six quantities r_1, r_2, r_3, r_4, α_1, α_2. Regarding the four sides and α_1 as given, this relation then determines α_2, and then α_3 and α_4 by means of equations (7). For this purpose equation (10) is most simply expressed as a cubic in β_2, namely,

$$\begin{aligned}(11)\quad &[\alpha_1^2 r_2^2 - \beta_1^2 r_4^2]\beta_2^3 + [\alpha_1^2\beta_1(r_1^2 - r_3^2) + \alpha_1^2 r_2^2 - \alpha_1\beta_1 r_4^2]\beta_2^2 \\ &\qquad - [\alpha_1^2(\alpha_1 + \beta_1)r_3^2]\beta_2 - \alpha_1^3 r_3^2 = 0.\end{aligned}$$

Thus if the four sides $r_1, \cdots, r_4$ are given, there may be three quadrilaterals which have the same value of α_1, but when a choice of these three has been made, r_5^2 and r_6^2 are simply computed by means of (8) and (9).

7. **The resultant acceleration.** The resultant acceleration of each of the particles m_1, m_2, m_3, and m_4 in turn due to the attraction of the other three particles is

$$\text{of } m_1, \qquad \frac{m_2}{r_1^3} r_1 a_1 + \frac{m_3}{r_5^3} I_{13} - \frac{m_4}{r_4^3} r_4 a_4 = A_1,$$

$$\text{of } m_2, \qquad \frac{m_3}{r_2^3} r_2 a_2 + \frac{m_4}{r_6^3} I_{24} - \frac{m_1}{r_1^3} r_1 a_1 = A_2,$$

$$\text{of } m_3, \qquad \frac{m_4}{r_3^3} r_3 a_3 + \frac{m_1}{r_5^3} I_{31} - \frac{m_2}{r_2^3} r_2 a_2 = A_3,$$

$$\text{of } m_4, \qquad \frac{m_1}{r_4^3} r_4 a_4 + \frac{m_2}{r_6^3} I_{42} - \frac{m_3}{r_3^3} r_3 a_3 = A_4.$$

On substituting the values of I_{ij} from (4) and using the notation

$$\frac{1}{r_i^3} = R_i,$$

these expressions become

$$\begin{aligned}(12)\quad A_1 &= \frac{1}{\alpha_2}[(\alpha_2 R_1 m_2 + \beta_2 R_5 m_3) r_1 a_1 + (\beta_4 R_5 m_3 - \alpha_2 R_4 m_4) r_4 a_4],\\ A_2 &= \frac{1}{\alpha_3}[(\alpha_3 R_2 m_3 + \beta_3 R_6 m_4) r_2 a_2 + (\beta_1 R_6 m_4 - \alpha_3 R_1 m_1) r_1 a_1];\\ A_3 &= \frac{1}{\alpha_4}[(\alpha_4 R_3 m_4 + \beta_4 R_5 m_1) r_3 a_3 + (\beta_2 R_5 m_1 - \alpha_4 R_2 m_2) r_2 a_2],\\ A_4 &= \frac{1}{\alpha_1}[(\alpha_1 R_4 m_1 + \beta_1 R_6 m_2) r_4 a_4 + (\beta_3 R_6 m_2 - \alpha_1 R_3 m_3) r_3 a_3].\end{aligned}$$

In order that the resulting acceleration on each particle may pass through the center of gravity of the system and be proportional to the distance of the particle from the center of gravity, it is necessary that

$$A_1 : A_2 : A_3 : A_4 :: G_1 : G_2 : G_3 : G_4.$$

Let the factor of proportionality be MR_0, so that

$$A_i - MR_0G_i = 0.$$

Furthermore, let

$$R_i - R_0 = S_i. \tag{13}$$

Then, on multiplying equations (6) by R_0 and subtracting from the corresponding equation of (12), it is found that the conditions which are necessary for a permanent configuration are

$$\begin{aligned}
&(\alpha_2S_1m_2 + \beta_2S_5m_3)r_1\boldsymbol{a}_1 + (\beta_4S_5m_3 - \alpha_2S_4m_4)r_4\boldsymbol{a}_4 = 0,\\
&(\alpha_3S_2m_3 + \beta_3S_6m_4)r_2\boldsymbol{a}_2 + (\beta_1S_6m_4 - \alpha_3S_1m_1)r_1\boldsymbol{a}_1 = 0,\\
&(\alpha_4S_3m_4 + \beta_4S_5m_1)r_3\boldsymbol{a}_3 + (\beta_2S_5m_1 - \alpha_4S_2m_2)r_2\boldsymbol{a}_2 = 0,\\
&(\alpha_1S_4m_1 + \beta_1S_6m_2)r_4\boldsymbol{a}_4 + (\beta_3S_6m_2 - \alpha_1S_3m_3)r_3\boldsymbol{a}_3 = 0.
\end{aligned} \tag{14}$$

Of course the masses must be positive and, since

$$MR_0 = \frac{M}{r_0^3} = \omega^2,$$

R_0 must be positive if the forces are attractive.

8. **The equations of condition.** If the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_4$ are noncollinear, it is necessary that all of the coefficients in equations (14) vanish, and since, by (7),

$$\frac{\beta_{i+1}}{\alpha_{i-1}} = -\frac{\alpha_i}{\beta_i} \qquad (i = 1, 2, 3, 4)$$

(circular permutation of the subscripts), this requires that

$$\begin{aligned}
&S_1\alpha_2m_2 + S_5\beta_2m_3 = 0, \qquad && S_6\alpha_2m_2 + S_3\beta_2m_3 = 0,\\
&S_2\alpha_3m_3 + S_6\beta_3m_4 = 0, && S_5\alpha_3m_3 + S_4\beta_3m_4 = 0,\\
&S_3\alpha_4m_4 + S_5\beta_4m_1 = 0, && S_6\alpha_4m_4 + S_1\beta_4m_1 = 0,\\
&S_4\alpha_1m_1 + S_6\beta_1m_2 = 0, && S_5\alpha_1m_1 + S_2\beta_1m_2 = 0.
\end{aligned} \tag{15}$$

In order to facilitate comparison, the equations in the second column have been circularly permuted once, that is, the last equation as derived from (14) has been placed first.

These equations are linear and homogeneous in the masses. A comparison of the first equations in each column shows that, since the determinant must vanish,

$$S_1S_3 = S_5S_6;$$

and from the other equations, taken in pairs, it is seen to be necessary that

$$S_1S_3 = S_2S_4 = S_5S_6. \tag{16}$$

If these conditions, equations (16), are satisfied, the equations in the second column of (15) will be satisfied if the equations in the first column are satisfied.

The determinant of the equations in the first column is easily found to be

$$\Delta = S_1S_2S_3S_4\alpha_1\alpha_2\alpha_3\alpha_4 - S_5^2S_6^2\beta_1\beta_2\beta_3\beta_4,$$

which, by virtue of equations (16), reduces to

$$\Delta = S_1S_2S_3S_4(\alpha_1\alpha_2\alpha_3\alpha_4 - \beta_1\beta_2\beta_3\beta_4).$$

From the first and third of equations (7) it is seen that

$$\alpha_1\alpha_2 = -\beta_2\beta_3,$$
$$\alpha_3\alpha_4 = -\beta_4\beta_1.$$

Hence

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4,$$

and the determinant vanishes, if equations (16) are satisfied. Three of the masses can then be determined in terms of the fourth; for example, from (15)

$$m_2 = -\frac{\alpha_1}{\beta_1}\frac{S_4}{S_6}m_1, \quad m_3 = -\frac{\beta_3}{\beta_1}\frac{S_1}{S_2}m_1, \quad m_4 = +\frac{\alpha_3}{\beta_1}\frac{S_1}{S_6}m_1. \tag{17}$$

9. **The necessary condition.** In order that the problem may admit a solution other than the straight line solution, it is necessary that

$$S_1S_3 = S_2S_4 = S_5S_6,$$

or

$$\begin{aligned}(R_1 - R_0)(R_3 - R_0) &= (R_2 - R_0)(R_4 - R_0)\\ &= (R_5 - R_0)(R_6 - R_0).\end{aligned}$$

From these equations it is found that

$$\begin{aligned} R_0 &= \frac{R_1R_3 - R_2R_4}{R_1 + R_3 - R_2 - R_4} = \frac{R_2R_4 - R_5R_6}{R_2 + R_4 - R_5 - R_6}\\ &= \frac{R_5R_6 - R_1R_3}{R_5 + R_6 - R_1 - R_3}, \end{aligned} \tag{18}$$

and therefore

$$
(19)\quad
\begin{aligned}
S_1 &= R_1 - R_0 = \frac{(R_1 - R_2)(R_1 - R_4)}{R_1 + R_3 - R_2 - R_4} = \frac{(R_1 - R_5)(R_1 - R_6)}{R_1 + R_3 - R_5 - R_6},\\
S_2 &= R_2 - R_0 = \frac{(R_2 - R_1)(R_3 - R_2)}{R_1 + R_3 - R_2 - R_4} = \frac{(R_2 - R_5)(R_2 - R_6)}{R_2 + R_4 - R_5 - R_6},\\
S_3 &= R_3 - R_0 = \frac{(R_3 - R_2)(R_3 - R_4)}{R_1 + R_3 - R_2 - R_4} = \frac{(R_3 - R_5)(R_3 - R_6)}{R_1 + R_3 - R_5 - R_6},\\
S_4 &= R_4 - R_0 = \frac{(R_3 - R_4)(R_4 - R_1)}{R_1 + R_3 - R_2 - R_4} = \frac{(R_4 - R_5)(R_4 - R_6)}{R_2 + R_4 - R_5 - R_6},\\
S_5 &= R_5 - R_0 = \frac{(R_1 - R_5)(R_5 - R_3)}{R_1 + R_3 - R_5 - R_6} = \frac{(R_2 - R_5)(R_5 - R_4)}{R_2 + R_4 - R_5 - R_6},\\
S_6 &= R_6 - R_0 = \frac{(R_1 - R_6)(R_6 - R_3)}{R_1 + R_3 - R_5 - R_6} = \frac{(R_2 - R_6)(R_6 - R_4)}{R_2 + R_4 - R_5 - R_6}.
\end{aligned}
$$

From these expressions for R_0 it is seen that if two pairs of opposite sides are given, R_0 is determined uniquely except when the members of one pair are equal respectively to the members of the other pair. Suppose r_1, r_2, r_3, and r_4 are given and that the two members of the pair r_1, r_3 are not equal to the two members of the pair r_2, r_4. Then

$$R_0 = R_1R_3 - R_2R_4/R_1 + R_3 - R_2 - R_4,$$

and, automatically,

$$S_1S_3 = S_2S_4 = \lambda,$$

where λ is some definite number. It is still necessary to determine the members of the other pair r_5, r_6, which may be called the diagonals, so that also

$$S_5S_6 = \lambda.$$

For this purpose the shape of the quadrilateral is available.

Before going into this, however, it is desirable to take up the exceptional case first, the case in which equality exists between the members of the two given pairs of sides.

10. **Particular case, $r_1 = r_2$, $r_3 = r_4$.** In the particular case in which $r_1 = r_2$ and $r_3 = r_4$ it can be assumed that $r_1 \geqq r_3$, as this is merely a matter of notation. The relation

$$S_1S_3 = S_2S_4$$

is satisfied whatever R_0 may be. Consequently R_0 can be chosen so that

$$S_1S_3 = S_2S_4 = S_5S_6,$$

that is, by using the second or third form of R_0 in equation (18).

It is seen from Fig. 1 that, for this case,

$$r_2\alpha_2 = -r_1\beta_1,$$

and

$$r_3\alpha_3 = -r_4\beta_4,$$

so that

$$\alpha_2 = -\beta_1, \qquad \alpha_3 = -\beta_4, \qquad \beta_2 = -\alpha_1, \qquad \beta_3 = -\alpha_4.$$

Then, by means of equations (7), it is found that

$$(20) \qquad \begin{aligned} \alpha_2 &= 1 - \alpha_1, & \beta_2 &= -\alpha_1, \\ \alpha_3 &= 2 - \alpha_1, & \beta_3 &= 1 - \alpha_1, \\ \alpha_4 &= \alpha_1 - 1, & \beta_4 &= \alpha_1 - 2, \end{aligned}$$

and all of the α's and β's are expressed simply in terms of α_1.

The expressions for the masses become

$$(21) \qquad \begin{aligned} m_2 &= -\frac{\alpha_1}{\beta_1}\frac{S_4}{S_6}m_1 = -\frac{\alpha_1}{\beta_1}\frac{R_5 - R_3}{R_1 - R_6}m_1, \\ m_3 &= m_1, \\ m_4 &= +\frac{2-\alpha_1}{\alpha_1 - 1}\frac{S_1}{S_6}m_1 = \frac{2-\alpha_1}{1-\alpha_1}\frac{R_5 - R_1}{R_6 - R_3}m_1; \end{aligned}$$

and for the diagonals

$$r_5^2 = \frac{\alpha_1^2}{\alpha_1 - 1}r_1^2 - \frac{(2-\alpha_1)^2}{\alpha_1 - 1}r_3^2, \quad r_6^2 = (1-\alpha_1)(r_1^2 - r_3^2).$$

Under the assumption that $r_1 \geqq r_3$, it is evident from the geometry that

$$r_1 - r_3 \leqq r_6 \leqq r_1 + r_3,$$

and, if ρ denotes the ratio of r_3 to r_1, it follows from the above expression for r_6^2 that

$$-2\rho/(1-\rho) \leqq \alpha_1 \leqq 2\rho/(1+\rho) \leqq +1.$$

It also follows that $\alpha_1 = 0$ when the sides r_3 and r_4 form a straight line; but this was already known from the definition of α_1.

11. Conditions necessary for positive masses. In the particular case under discussion, m_3 is positive if m_1 is positive, which will be assumed. Starting with the maximum value of α_1, namely

$$\alpha_1 = 2\rho/(1+\rho) \leqq 1,$$

for which r_5 vanishes and $r_6 = r_1 - r_3$, the ratio α_1/β_1 is negative until α_1 vanishes and thereafter is positive, but the coefficient $(2-\alpha_1)/(1-\alpha_1)$ in m_4

is always positive. Changes of sign in the masses m_2 and m_4, considered as functions of α_1 and ρ, occur for the following critical values (compare (21)):

$$\text{(a)} \qquad r_5 = r_3 \quad \text{for which} \quad \rho = \frac{\pm\, \alpha_1}{(3 - 3\alpha_1 + \alpha_1^2)^{1/2}}, \quad m_2 = 0,$$

$$\text{(b)} \qquad r_5 = r_1 \quad \text{for which} \quad \rho = \frac{(1 - \alpha_1 + \alpha_1^2)^{1/2}}{2 - \alpha_1}, \quad m_4 = 0,$$

$$\text{(c)} \qquad \alpha_1 = 0, \quad m_2 = 0,$$

$$\text{(d)} \qquad r_6 = r_3 \quad \text{for which} \quad \alpha_1 = \frac{1 - 2\rho^2}{1 - \rho^2}, \quad m_4 = \infty,$$

$$\text{(e)} \qquad r_6 = r_1 \quad \text{for which} \quad \alpha_1 = \frac{-\,\rho^2}{1 - \rho^2}, \quad m_2 = \infty.$$

The curves represented by these equations are shown in Fig. 2, in which α is the abscissa and ρ is the ordinate. The area in which m_2 is positive is hatched horizontally. The area in which m_4 is positive is hatched obliquely. Consequently both m_2 and m_4 are positive in the area that is cross hatched. Any point in this cross hatched area leads to a real, positive solution of the problem, provided R_0 also is positive, as is actually the case. It is seen that there are two such areas that are not connected, and that one of these areas is sub-divided into two areas that are connected at the point $\alpha_1 = +1/2$,

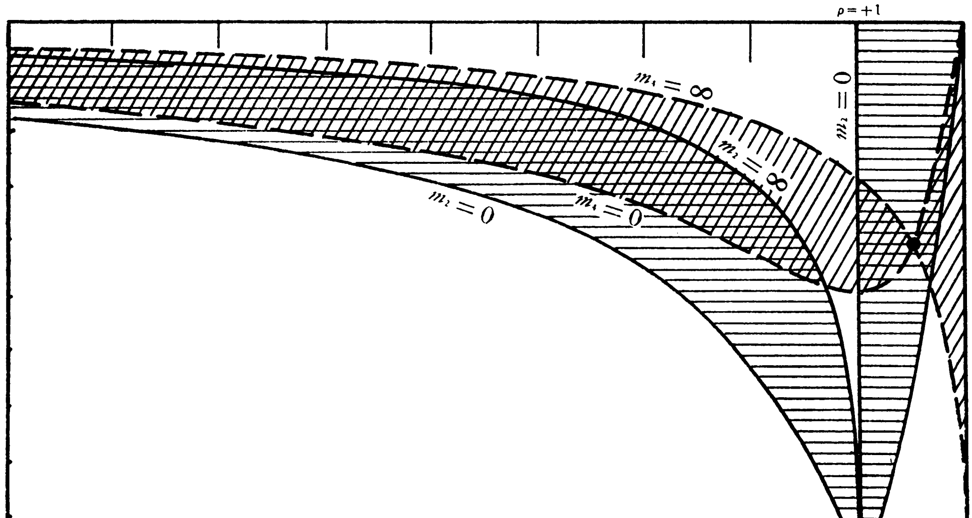

Fig. 2

$\rho = 1/3^{1/2}$. This point corresponds to the equilateral triangle in which $r_1 = r_2 = r_5$, and since $r_3 = r_6$, the fourth particle m_4 is at the center of the equilateral triangle. Three of the masses are arbitrary but equal, while the fourth is entirely arbitrary, and this is the only case in which the masses are not uniquely determined, if the quadrilateral is given.

The condition that $R_0 = 0$ is

$$R_1R_3 - R_5R_6 = 0,$$

and this leads to the equation

$$\rho^2 = \frac{2\alpha_1^2 - 4\alpha_1 + 3 \pm (3(2\alpha_1 - 3)(2\alpha_1 - 1))^{1/2}}{2(2 - \alpha_1)^2}.$$

The condition that $R_0 = \infty$ is

$$R_1 + R_3 - R_5 - R_6 = 0.$$

It can be expressed as an equation in ρ and α_1 but is too complicated to write down. Both of the curves $R_0 = 0$ and $R_0 = \infty$ are shown in Fig. 3. The two

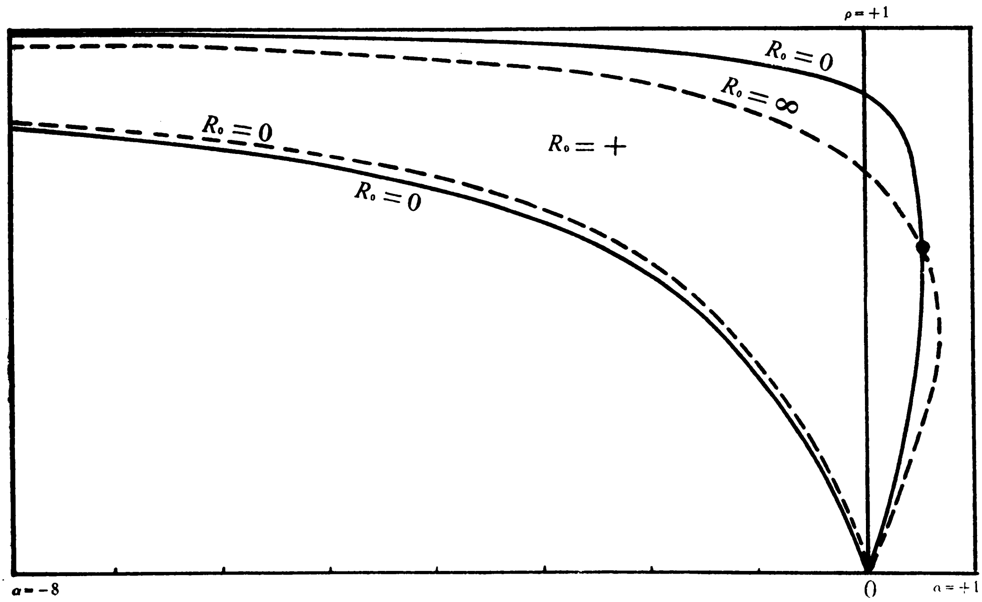

Fig. 3

curves intersect at the points 0, 0 and 1/2, $1/3^{1/2}$, both of which are conspicuous in Fig. 2. A superposition of Figs. 2 and 3 shows that R_0 is positive everywhere within the cross hatched area of Fig. 2. There are therefore two groups of solutions for the special problem in which $r_1 = r_2$. In the first group α_1 is negative, and the quadrilateral is convex. In the second group

$$0 \leqq \alpha_1 \leqq +1,$$

and the quadrilateral is concave. The masses m_1 and m_3 are always equal, while m_2 and m_4 may have any positive values whatever.

12. **Resumption of the general case.** Aside from the exceptional case in which two pairs of adjacent sides are equal, R_0 is uniquely determined by the condition (§9)

$$S_1S_3 = S_2S_4 = \lambda,$$

assuming that r_1, r_2, r_3 and r_4 are given. The shape of the quadrilateral, however, is still arbitrary, that is, α_1 is still at our disposal. After α_1 has been fixed α_2 is determined by equation (11), which is a cubic in β_2, and then α_3 and α_4 by equations (7). Hence r_5 and r_6 can be thought of as functions of α_1, and the point is to determine α_1 so that

$$S_5S_6 = \lambda.$$

From the first equation of (8) and the second equation of (9) are obtained

$$\alpha_3 r_3 = \pm (\alpha_1\beta_1 r_1^2 + \alpha_1 r_2^2 - \beta_1 r_5^2)^{1/2} = Q_1,$$

and

$$\beta_3 r_3 = \pm (\alpha_1 r_6^2 + \alpha_1\beta_1 r_1^2 - \beta_1 r_4^2)^{1/2} = Q_2.$$

Using the first of these equations, it is found that

$$\alpha_2 = \frac{-r_3 + Q_1}{\beta_1 r_3 + Q_1}, \qquad \alpha_3 = \frac{Q_1}{r_3}, \qquad \alpha_4 = \frac{\beta_1 r_3}{\beta_1 r_3 + Q_1},$$

$$\beta_2 = \frac{-\alpha_1 r_3}{\beta_1 r_3 + Q_1}, \qquad \beta_3 = \frac{-r_3 + Q_1}{r_3}, \qquad \beta_4 = \frac{-Q_1}{\beta_1 r_3 + Q_1};$$

and from the second equation,

$$\alpha_2 = \frac{Q_2}{\alpha_1 r_3 + Q_2}, \qquad \alpha_3 = 1 + \frac{Q_2}{r_3}, \qquad \alpha_4 = \frac{\beta_1 r_3}{\alpha_1 r_3 + Q_2},$$

$$\beta_2 = \frac{-\alpha_1 r_3}{\alpha_1 r_3 + Q_2}, \qquad \beta_3 = \frac{Q_2}{r_3}, \qquad \beta_4 = -\frac{r_3 + Q_2}{\alpha_1 r_3 + Q_2}.$$

On substituting these results in the second equation of (8) and the first equation of (9), the two following equations are derived:

$$\begin{aligned} -\beta_1 r_5^4 &+ [\alpha_1^2 r_1^2 + \alpha_1(-r_1^2 + r_2^2 - r_3^2 + r_4^2) + (r_3^2 - r_4^2)]r_5^2 \\ &+ [\alpha_1^2 r_1^2(r_3^2 - r_4^2) + \alpha_1(r_3^2 - r_4^2)(r_2^2 - r_1^2)] \\ &+ [\alpha_1(r_1^2 - r_2^2 + r_5^2) - 2r_5^2]r_3 Q_1 = 0, \end{aligned} \tag{22}$$

and

$$
\begin{aligned}
&+ \alpha_1 r_6^4 + [\alpha_1^2 r_1^2 + \alpha_1(-r_1^2 - r_2^2 + r_3^2 - r_4^2) + r_4^2] r_6^2 \\
(23) \qquad &+ [-\alpha_1^2(r_2^2 - r_3^2) r_1^2 + \alpha_1(r_1^2 + r_4^2)(r_2^2 - r_3^2) - (r_2^2 - r_3^2) r_4^2] \\
&+ [\alpha_1(r_1^2 - r_4^2 + r_6^2) + (-r_1^2 + r_4^2 + r_6^2)] r_3 Q_2 = 0.
\end{aligned}
$$

If rationalized the first would be an equation of the eighth degree in r_5 and the second of the eighth degree in r_6. These equations determine r_5 and r_6 as functions of α_1 and the four sides r_1, r_2, r_3, and r_4. On substituting these values of r_5 and r_6 in the equation

$$\left(\frac{1}{r_5^3} - \frac{1}{r_0^3}\right)\left(\frac{1}{r_6^3} - \frac{1}{r_0^3}\right) = \lambda$$

there is derived an equation which determines α_1. It is not practical to do this literally. One can determine α_1 as accurately as may be desired from these equations by a series of approximations in numerical cases, but the process is laborious.

13. **Convex and concave quadrilaterals.** Suppose that pegs are placed at each of the four masses. A string is passed around them and drawn taut. Two cases are distinguishable:

I. The string touches all four pegs, and
(a) no three pegs are in a straight line;
(b) three pegs are in a straight line, but not four;
(c) four pegs are in a straight line.

II. The string touches only three pegs, the fourth being inside of the triangle formed by the other three.

If the conditions of Case I(a) or I(b) are satisfied, the quadrilateral will be called *convex*. Case I(c) is the straight line configuration with which we are not concerned. If the conditions of Case II are satisfied, the quadrilateral will be called *concave*.

In all cases at least three masses touch the string. Let these three masses in counterclockwise order be m_1, m_2, and m_3. If m_4 also lies on the string it is between m_3 and m_1 and the quadrilateral is convex. If it does not touch the string it lies inside of the triangle formed by m_1, m_2, and m_3, and the quadrilateral is concave. This convention as to the masses eliminates duplication of cases that differ essentially in notation only.

It is seen from Fig. 1 that the ratio

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_1}{\alpha_1 - 1}$$

is positive wherever the point B on the line L_1 may be, provided it does not lie between m_1 and m_2, and in the interval m_1m_2 it is negative. In general,

(24) for convex quadrilaterals, $\dfrac{\alpha_1}{\beta_1} > 0, \quad \dfrac{\alpha_2}{\beta_2} > 0, \quad \dfrac{\alpha_3}{\beta_3} > 0, \quad \dfrac{\alpha_4}{\beta_4} > 0;$

and

(25) for concave quadrilaterals, $\dfrac{\alpha_1}{\beta_1} < 0, \quad \dfrac{\alpha_2}{\beta_2} < 0, \quad \dfrac{\alpha_3}{\beta_3} > 0, \quad \dfrac{\alpha_4}{\beta_4} > 0.$

14. **Admissible convex quadrilaterals.** From equations (13), (15), and (16) one sees that in any solution of the problem

$$
\begin{aligned}
m_2 &= -\frac{\alpha_1}{\beta_1}\frac{R_4 - R_0}{R_6 - R_0}m_1 = -\frac{\alpha_1}{\beta_1}\frac{R_5 - R_0}{R_2 - R_0}m_1,\\
m_3 &= -\frac{\alpha_2}{\beta_2}\frac{R_1 - R_0}{R_5 - R_0}m_2 = -\frac{\alpha_2}{\beta_2}\frac{R_6 - R_0}{R_3 - R_0}m_2,\\
m_4 &= -\frac{\alpha_3}{\beta_3}\frac{R_2 - R_0}{R_6 - R_0}m_3 = -\frac{\alpha_3}{\beta_3}\frac{R_5 - R_0}{R_4 - R_0}m_3,\\
m_1 &= -\frac{\alpha_4}{\beta_4}\frac{R_3 - R_0}{R_5 - R_0}m_4 = -\frac{\alpha_4}{\beta_4}\frac{R_6 - R_0}{R_1 - R_0}m_4;
\end{aligned}
\tag{26}
$$

and

$$(R_1 - R_0)(R_3 - R_0) = (R_2 - R_0)(R_4 - R_0) = (R_0 - R_5)(R_0 - R_6). \tag{27}$$

First hypothesis: $r_1 > r_0$. From their definitions it follows, if $r_1 > r_0$, that

$$R_1 < R_0;$$

and since for convex quadrilaterals $\alpha_i/\beta_i > 0$, $i = 1, 2, 3, 4$, and since the masses are necessarily positive, it is found from equations (26) that

$$R_1, R_2, R_3, R_4 < R_0 < R_5, R_6,$$

and therefore

$$r_1, r_2, r_3, r_4 > r_0 > r_5, r_6;$$

that is, each of the four sides is greater than r_0, and each of the two diagonals is less than r_0. This is a geometric absurdity for a convex quadrilateral, as is easily proved. Hence there are no solutions of the problem in which $r_1 > r_0$.

Second hypothesis: $r_1 \leqq r_0$. This hypothesis merely reverses the inequalities of the first hypothesis, but includes the equality sign. Hence, in this case,

$$r_1, r_2, r_3, r_4 \leqq r_0 \leqq r_5, r_6;$$

that is, each of the four sides is less than, or, at most, equal to r_0, and each of the diagonals is greater than r_0. Quadrilaterals of this type are geometrically possible. In order to show this, let r_0 and r_1 be given with

$$r_1 < r_0.$$

Draw r_1 as in Fig. 4, and let m_1 and m_2 be at its extremities. With m_1 and

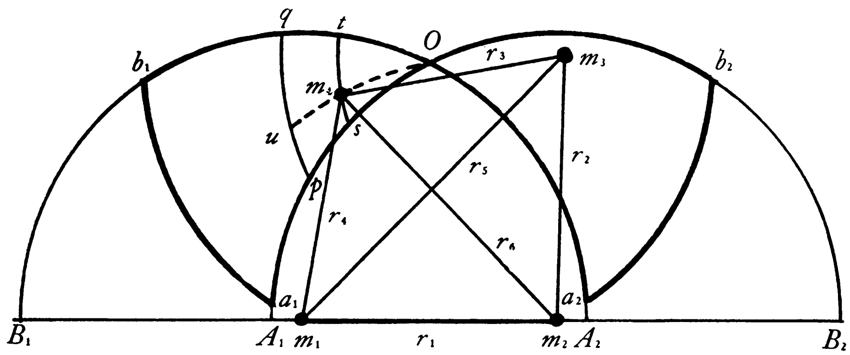

Fig. 4

m_2 as centers draw semicircles with the radius r_0, intersecting at the point O. With O as a center draw the arcs a_1b_1 and a_2b_2 also with the radius r_0. Since

$$r_5 > r_0 > r_2,$$

the mass m_3 lies outside of the semicircle B_1OA_2 and inside of the semicircle A_1OB_2; that is, it lies inside of the area OA_2B_2. Likewise, since

$$r_6 > r_0 > r_4,$$

the mass m_4 lies outside of the semicircle A_1OB_2 and inside of the semicircle B_1OA_2; that is, it lies inside of the area OA_1B_1. The possibilities are further restricted by the fact that r_3, which is the line joining m_3 and m_4, also is less than r_0. Hence m_3 must lie inside of the area Oa_2b_2, and m_4 must lie inside of the area Ob_1a_1; and the distance between m_3 and m_4 must be less than r_0.

A quadrilateral will be called an admissible quadrilateral if, for properly chosen masses, it can form a permanent configuration. With this definition we can state the following theorem:

THEOREM. *For every point m_3 in the region Oa_2b_2 there exists one and only one point m_4 in the region Ob_1a_1 which together with the points m_1 and m_2 forms an admissible convex quadrilateral; and for every point m_4 in the region Ob_1a_1 there exists one and only one point m_3 in the region Oa_2b_2 which together with the points m_1 and m_2 forms an admissible convex quadrilateral. No such points exist outside of the areas Ob_1a_1 and Oa_2b_2.*

Let m_3 be taken anywhere within the area Oa_2b_2. With m_3 as a center and a radius r_0 describe the arc pq intersecting the arcs Oa_1 and Ob_1 in p and q. Any point m_4 which lies in the area Opq together with m_1, m_2, and m_3 will form a quadrilateral which satisfies the inequalities

$$r_1,\ r_2,\ r_3,\ r_4 < r_0 < r_5,\ r_6.$$

It remains to show that there is one and only one such point at which the equalities

$$(28)\quad (R_1 - R_0)(R_3 - R_0) = (R_2 - R_0)(R_4 - R_0) = (R_0 - R_5)(R_0 - R_6)$$

also are satisfied. The quantities R_0, R_1, R_2, and R_5 are given by the assumed data.

All of the factors which occur in (28) are positive in the area Opq. With m_3 as a center and a radius ρ, such that

$$\overline{Om_3} \leqq \rho \leqq r_0,$$

describe an arc of a circle which cuts Op at the point s and Oq at the point t. Imagine the point m_4 lying on this arc st and moving from s to t. The factor $(R_4 - R_0)$ is positive at s, decreases steadily, and vanishes at t, while the factor $(R_0 - R_6)$ vanishes at s, and, increasing steadily, is positive at t. Hence there exists one and only one point p on st at which the equality

$$(R_2 - R_0)(R_4 - R_0) = (R_0 - R_5)(R_0 - R_6) = \lambda(\rho)$$

is satisfied. This is true for every value of ρ, and the locus of p as ρ increases is a certain curve C which passes through the point O and cuts the arc pq in some point u. It is evident that $\lambda(\rho)$ vanishes at O. Its derivative with respect to ρ is

$$(29)\qquad \frac{d\lambda}{d\rho} = (R_2 - R_0)\frac{dR_4}{d\rho} = -(R_0 - R_5)\frac{dR_6}{d\rho}\,.$$

Since $R_i = 1/r_i^3$,

$$\frac{dR_4}{d\rho} > 0 \quad\text{and}\quad \frac{dR_6}{d\rho} < 0,$$

and therefore

$$\frac{d\lambda}{d\rho} > 0,$$

as the point p moves along the curve C_1 at O. Regarding r_4 and r_6 as bipolar coördinates of the point p , it is seen that dr_4 and dr_6 cannot both vanish, since the point p does not lie in the line which passes through m_1 and m_2.

Therefore dR_4 and dR_6 cannot vanish simultaneously, and since (R_2-R_0) and (R_0-R_5) are constants, it is seen from (29) that neither can vanish, and therefore $d\lambda/d\rho$ is always positive. Hence the value of λ increases steadily from zero at O to some positive value at u.

On the other hand the value of $(R_1-R_0)(R_3-R_0)$ is positive at O, decreases steadily, and vanishes at u. Hence, for a given m_3, there exists one and only one point m_4 on the curve C, and therefore within the area Opq, at which the equalities

$$(R_1 - R_0)(R_3 - R_0) = (R_2 - R_0)(R_4 - R_0) = (R_0 - R_5)(R_0 - R_6)$$

are satisfied.

The first half of the theorem as stated is therefore established; and the second half follows from symmetry.

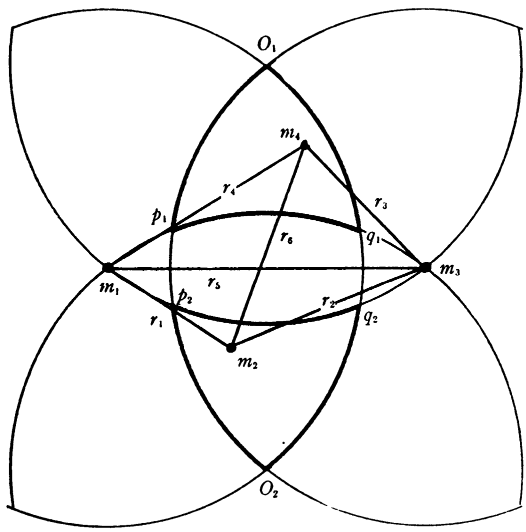

Fig. 5

15. **The ratio of the diagonals.** Instead of drawing the diagram with r_1 and r_0 as the fundamental lines, let r_5 and r_0 be used $(r_5>r_0)$. In Fig. 5, let m_1 and m_3 be the end points of r_5. With each of these points as a center draw circles of radius r_0 intersecting at the points O_1 and O_2. With the points O_1 and O_2 as centers draw two more circles with the radius r_0. These last two circles pass through m_1 and m_3 and intersect the arcs O_1O_2 in the points p_1, q_1; p_2, and q_2.

Since for all admissible quadrilaterals

$$r_1,\ r_2,\ r_3,\ r_4 \leqq r_0 \leqq r_5,\ r_6,$$

it is seen that m_4 lies in the area $O_1p_1q_1$, and m_3 lies in the area $O_2p_2q_2$, and it can be shown, just as before, that for every m_4 in the area $O_1p_1q_1$ there exists one and only one m_2 in the area $O_2p_2q_2$ which, together with m_1, m_3, and m_4, forms an admissible quadrilateral. Similarly, for each m_2 in $O_2p_2q_2$ there is one and only one m_4 in the area $O_1p_1q_1$.

It is evident that if r_0 is kept fixed, and r_1 is increased, the areas $O_ip_iq_i$ shrink and for a certain value of r_1 are reduced to two points. For larger values of r_1 admissible convex quadrilaterals do not exist.

From Fig. 5 is obtained

$$\left(\frac{r_5}{2}\right)^2 + \left(\frac{O_1O_2}{2}\right)^2 = r_0^2.$$

But since $\overline{O_1O_2} \geqq r_6 \geqq r_0$, it follows that

$$r_5 \leqq 3^{1/2}r_0, \quad \text{and} \quad r_6 \leqq 3^{1/2}r_0;$$

therefore

$$r_5 \leqq 3^{1/2}r_6, \quad \text{and} \quad r_6 \leqq 3^{1/2}r_5.$$

Whence the

THEOREM. *The ratio of the diagonals of an admissible convex quadrilateral lies between* $1/3^{1/2}$ *and* $3^{1/2}$.

This theorem is a generalization of Longley's theorem for the rhombus.

16. Limitations on the interior angles. There is a corresponding limitation on the magnitudes of the interior angles of an admissible convex quadrilateral. It is evident from Fig. 5 that the interior angle at m_1, $\angle m_1$, is less than the angle $O_1m_1O_2$; and that the maximum value of this latter angle is had at the limit $r_1 = r_0$, in which case it is 120°. Hence

$$\angle\, m_1 \leqq 120^\circ.$$

From Fig. 4 it is also evident that the $\angle m_1$ is greater than the angle Om_1m_2, and that the minimum value of $\angle Om_1m_2$ is had for the limiting value $r_1 = r_0$ in which case it is 60°. Combining these two results, we have

$$60^\circ \leqq \angle\, m_1 \leqq 120^\circ.$$

The same inequality holds obviously for $\angle m_2$. On interchanging the rôle of m_4 and m_1, and m_3 and m_2, it is seen that the same inequality holds for all of the interior angles.

THEOREM. *Each of the interior angles of an admissible convex quadrilateral lies between* 60° *and* 120°.

By a similar method, using Fig. 5, it can be shown that we have the

Theorem. *In any admissible convex quadrilateral the diagonals divide each of the interior angles into two angles each of which is less than* 60°.

17. **Admissible concave quadrilaterals.** If $r_1 < r_0$, it is found from equations (25) and (26) that

$$r_3,\ r_4,\ r_6 > r_0 > r_1,\ r_2,\ r_5$$

which is geometrically impossible for a concave quadrilateral.

If $r_1 \geqq r_0$, the inequality signs are reversed and the equality sign is added. Hence a necessary condition for admissible concave quadrilaterals is that

$$r_3,\ r_4,\ r_6 \leqq r_0 \leqq r_1,\ r_2,\ r_5. \tag{30}$$

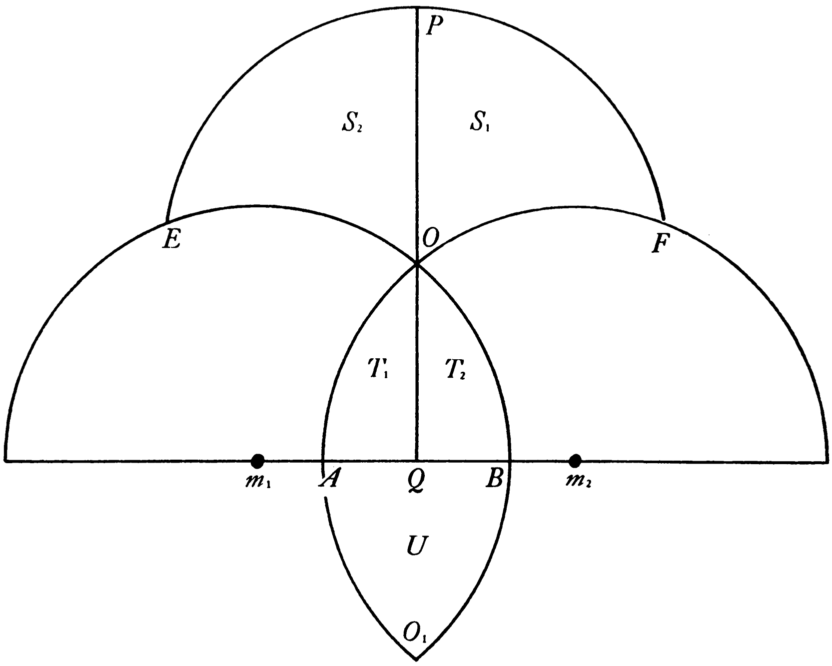

Fig. 6

The possibility of quadrilaterals of this type is shown in Fig. 6. Let $\overline{m_1m_2}$ be r_1. With m_1 and m_2 as centers draw circles of radius r_0. Since $r_4+r_6 \geqq r_1 \geqq r_0$ and $r_4,\ r_6 \leqq r_0$, it is seen that

$$r_0 \leqq r_1 \leqq 2r_0.$$

Let these two circles intersect in the points O and O_1. With O as a center and a radius equal to r_0 draw the arc EPF.

On account of the inequalities (30) and the adopted conventions (§13) it is evident that m_4 must lie in the region T_1 or T_2, and m_3 must lie in S_1 or S_2. An argument similar to that used in §14 shows that if m_4 is any assigned point in T_1+T_2 there exists one and only one point m_3 in the region S_1+S_3; but for any assigned point m_3 in the region S_1+S_3 there exists one and only one point m_4 in the region T_1+T_2+U, at which the equalities

$$\tag{31} \begin{aligned}(R_0 - R_1)(R_3 - R_0) &= (R_0 - R_2)(R_4 - R_0)\\ &= (R_0 - R_5)(R_6 - R_0)\end{aligned}$$

are satisfied. All of the factors in (31) are positive if m_3 and m_4 lie in the assigned regions. One sees from the last equality, if $r_2 < r_5$, that $r_4 < r_6$, and therefore if m_4 lies in T_1, m_3 lies in S_1; and if m_4 lies in T_2, m_3 lies in S_2. Notwithstanding the fact that for all admissible concave quadrilaterals m_4 lies in T_1+T_2 and m_3 in S_1+S_2, it is not true that all quadrilaterals which satisfy this condition are concave. There exist convex quadrilaterals for which m_4 lies in T_1+T_2 and m_3 in S_1+S_2 for which the equalities are satisfied, but for all such convex quadrilaterals at least one of the masses is negative, as was shown in §14. This means that the regions S_1+S_2 and T_1+T_2 are not sufficiently restricted. A plane concave quadrilateral cannot change in a continuous manner into a convex quadrilateral without passing through a configuration in which three of the corners lie in a straight line, and the curves on which this happens pass through the regions T_1, T_2, S_1, and S_2. Since the masses are all positive for the concave quadrilaterals and at least one of the masses negative for the convex quadrilaterals, it follows that at least one of the masses vanishes on the boundary.

There are three of these bounding loci:

(I) m_4 lies on the line m_1m_2,

(II) m_4 lies on the line m_1m_3,

(III) m_4 lies on the line m_2m_3.

The two dynamical equations (31) must be satisfied in all cases, but the geometric equations (8) and (9) take different forms in the different cases.

Case I. *m_4 lies between m_1 and m_2.* One finds readily under this condition that

$$\alpha_1 = \frac{r_4}{r_1}, \qquad \alpha_2 = 0, \qquad \alpha_3 = +1,$$

$$\alpha_4 = 1 - \frac{r_1}{r_4} = -\frac{r_6}{r_4},$$

and the geometric equations (8) and (9) reduce to the two equations

$$r_1 = r_4 + r_6,$$

$$r_1 r_3^2 = r_4 r_2^2 + r_6 r_5^2 - r_1 r_4 r_6.$$

Thus between the five quantities r_2, r_3, r_4, r_5, and r_6 there exist four relations. One can imagine r_3, r_5, and r_6 eliminated between these four equations, leaving a single relation which defines the locus sought by means of the bipolar coördinates r_2 and r_4. Actually it is not practical to do this, and one must resort to methods of successive approximations in numerical cases in order to obtain points on the curves. It should be noticed that these curves depend upon the ratio r_0/r_1, and therefore cannot be drawn once for all.

It is easy to solve the equations if m_4 coincides with either of the points A or B of Fig. 6. The following are the results:

$$\text{at } A, \quad r_2 = r_3 = r_6 = r_0, \quad r_4 = r_1 - r_0,$$

$$r_5^2 = r_1^2 - r_1 r_0 + r_0^2;$$

$$\text{at } B, \quad r_3 = r_4 = r_5 = r_0, \quad r_6 = r_1 - r_0,$$

$$r_2^2 = r_1^2 - r_1 r_0 + r_0^2.$$

At the point Q, a fairly simple solution can be obtained, since

$$r_1 = 2r_4 = 2r_6,$$

and the necessary equations reduce to

$$(R_0 - R_1)(R_3 - R_0) = (R_0 - R_2)(8R_1 - R_0),$$

$$r_2^2 = r_3^2 + \tfrac{1}{4} r_1^2.$$

Let $r_3 = \lambda r_1$, so that $r_2^2 = (\lambda^2 + \frac{1}{4}) r_1^2$. Also, if one takes $\lambda^{-3} = a$ and $(\lambda^2 + \frac{1}{4})^{-3/2} = b$, then

$$R_3 = aR_1, \qquad R_2 = bR_1,$$

and

$$R_1 = \frac{7 + b - a}{8b - a} R_0.$$

In order that the inequalities $r_0 < r_1 < 2r_1$ may hold it is readily found that

$$3^{1/2} > 2\lambda > 1.$$

The following table shows the values of the various ratios if m_4 is at the central point Q.

TABLE I. VALUES OF THE RATIOS AT Q

λ	$\frac{r_1}{r_0}=\frac{2r_4}{r_0}=\frac{2r_6}{r_0}$	$\frac{r_3}{r_0}$	$\frac{r_2}{r_0}$
.8667	1.000	.8667	1.000
.85	1.014	.8620	1.000
.80	1.067	.8536	1.006
.75	1.126	.8445	1.015
.70	1.195	.8365	1.028
.65	1.281	.8326	1.041
.64	1.301	.8326	1.056
.63	1.322	.8329	1.063
.62	1.344	.8333	1.070
.61	1.370	.8357	1.080
.60	1.396	.8376	1.090
.59	1.425	.8408	1.102
.58	1.458	.8456	1.116
.57	1.492	.8504	1.131
.56	1.533	.8584	1.151
.55	1.578	.8679	1.173
.54	1.631	.8807	1.200
.53	1.696	.8989	1.236
.52	1.772	.9214	1.278
.51	1.869	.9532	1.3314
.50	2.000	1.0000	1.4141
By interpolation	1.3	.8325	1.055
	1.4	.838	1.090
	1.5	.852	1.134

The following table gives values of the ratios on this curve for $r_1 = 1.5r_0$.

r_2/r_0	r_3/r_0	r_4/r_0	r_5/r_0
1.323	1.	1.	1.
1.301	.960	.950	1.006
1.250	.923	.900	1.019
1.195	.863	.800	1.080
1.134	.852	.750	1.134

Case II. *m_4 lies on the line between m_1 and m_3.* In addition to the dynamical equations, the geometrical equations are

$$r_5 = r_3 + r_4,\ r_5 r_6^2 = r_4 r_2^2 + r_3 r_1^2 - r_3 r_4 r_5.$$

Two points on the curve, the two end points, are easily obtained, viz.:

(*a*) $r_2 = r_3 = r_6 = r_0, \quad r_5 = r_4 + r_0, \quad r_0^2 + r_0 r_4 + r_4^2 = r_1^2;$

(*b*) $r_3 = r_4 = r_6 = r_0, \quad r_5 = 2r_0, \quad r_2^2 = 4r_0^2 - r_1^2.$

Since $r_2 \geqq r_0$, (30), it follows that the point b is not real unless $r_1 \leqq 3^{1/2}\ r_0$. That is, Case II offers no restriction on the areas T_1 and S_1, if $r_1 > 3^{1/2}\ r_0$.

The following table gives the values of the ratios at three points on this locus for $r_1 = 1.5\ r_0$.

r_2/r_0	r_3/r_0	r_4/r_0	r_6/r_0
1.000	1.000	.725	1.000
1.135	.950	.900	.958
1.323	1.000	1.000	1.000

Case III. *m_4 lies on the line between m_2 and m_3.* This case is symmetrical with Case II and gives the loci which pass through and restrict the regions S_2 and T_2.

Fig. 7 shows the reduced areas S_1, S_2, T_1, and T_2 for the case $r_1 = 1.5\ r_0$.

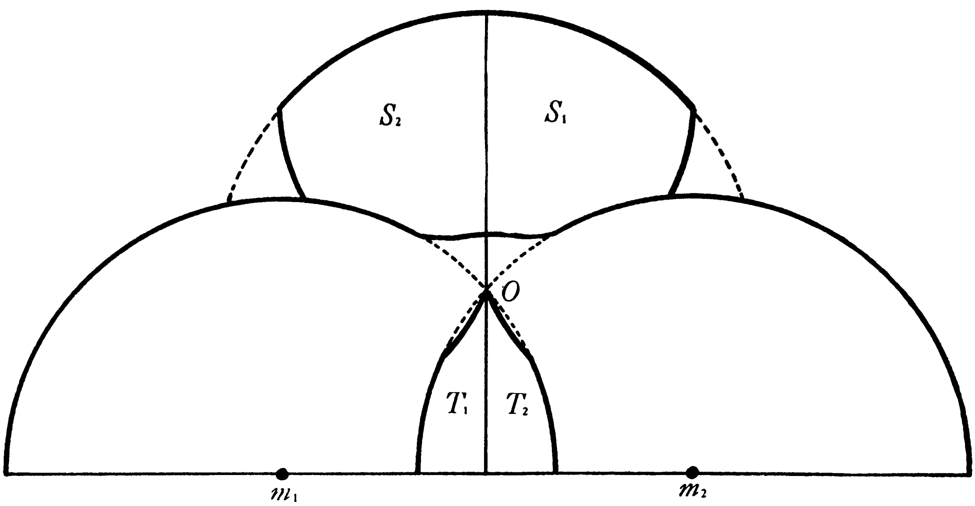

Fig. 7

18. **Isosceles trapezoids.** In case the convex quadrilateral is a trapezoid with r_3 parallel to r_1,

$$\alpha_1 = \beta_1 = \alpha_3 = \beta_3 = \infty;$$

and

$$\alpha_2 = -\beta_4, \qquad \alpha_4 = -\beta_2.$$

If the trapezoid is isosceles, one has also

$$r_2 = r_4, \qquad r_5 = r_6,$$

and

$$\alpha_2 = \frac{r_1}{r_1 - r_3}, \qquad \beta_2 = \frac{r_3}{r_1 - r_3}.$$

The second equation of (8), which is geometrical, reduces to

$$r_5^2 = r_1 r_3 + r_2^2. \tag{32}$$

The dynamical equation (31) becomes

$$(R_1 - R_0)(R_3 - R_0) = (R_2 - R_0)^2 = (R_0 - R_5)^2, \tag{33}$$

from the last equality of which it follows, on extracting the square root, that, since $r_2 < r_0 < r_5$ by §14,

$$\begin{aligned} R_2 - R_0 &= R_0 - R_5 > 0, \text{ or} \\ R_2 + R_5 &= 2R_0. \end{aligned} \tag{34}$$

A parametric solution of equations (32) and (33) can be obtained as follows. Define the parameter κ by the relation

$$r_2 = \kappa(r_1 r_3)^{1/2}, \tag{35}$$

and it follows from (32) that

$$r_5 = (1 + \kappa^2)^{1/2}(r_1 r_3)^{1/2}; \tag{36}$$

also (34) becomes

$$\frac{R_0}{(R_1 R_3)^{1/2}} = \tfrac{1}{2}\left\{\kappa^{-3} + (1 + \kappa^2)^{-3/2}\right\}. \tag{37}$$

From the first equality of (33) is obtained

$$R_1 R_3 - (R_1 + R_3)R_0 = R_2^2 - 2R_2 R_0,$$

which becomes, on using (35),

$$(1 - \kappa^{-6})R_1 R_3 + 2\kappa^{-3} R_0 (R_1 R_3)^{1/2} - (R_1 + R_3)R_0 = 0,$$

or

$$(1 - \kappa^{-6}) + 2\kappa^{-3}\frac{R_0}{(R_1 R_3)^{1/2}} - \left(\frac{R_0}{R_1} + \frac{R_0}{R_3}\right) = 0.$$

Then, by means of (37),

$$(1 - \kappa^{-6}) + \kappa^{-3}\left\{\kappa^{-3} + (1 + \kappa^2)^{-3/2}\right\} - \left(\frac{R_0}{R_1} + \frac{R_0}{R_3}\right) = 0. \tag{38}$$

From (37) and (38), it is found that

$$\frac{R_0}{R_1}+\frac{R_0}{R_3} = 1 + \kappa^{-3}(1+\kappa^2)^{-3/2} = 1 + ab,$$

$$\frac{2R_0}{(R_1R_3)^{1/2}} = \kappa^{-3} + (1+\kappa^2)^{-3/2} = a + b,$$

where

$$a = \kappa^{-3}, \qquad b = (1+\kappa^2)^{-3/2}.$$

The solution of these equations is

$$(39)\qquad \begin{aligned} r_1^3 &= \tfrac{1}{2}[1 + ab \pm ((1-a^2)(1-b^2))^{1/2}]r_0^3, \\ r_3^3 &= \tfrac{1}{2}[1 + ab \mp ((1-a^2)(1-b^2))^{1/2}]r_0^3, \end{aligned}$$

and adding

$$(40)\qquad \begin{aligned} r_2 &= \kappa(r_1r_3)^{1/2} = \left(\frac{a+b}{2a}\right)^{1/3} r_0, \\ r_5 &= (1+\kappa^2)^{1/2}(r_1r_3)^{1/2} = \left(\frac{a+b}{2b}\right)^{1/3} r_0, \\ a &= \kappa^{-3}, \quad b = (1+\kappa^2)^{-3/2}, \end{aligned}$$

the parametric representation is complete.

A table of values of the ratios of r_0, r_2, and r_5 to r_1 is given in the table below.

TABLE II

a	b	r_0/r_1	r_2/r_1	r_3/r_1	r_5/r_1
0	.0000	1.0000	1.0000	.0000	1.0000
.1	.0746	1.0001	.9556	.1968	1.0537
.2	.1286	1.0004	.9370	.3002	1.0855
.3	.1722	1.0015	.9246	.3831	1.1127
.4	.2087	1.0034	.9160	.4555	1.1378
.5	.2403	1.0066	.9106	.5224	1.1626
.6	.2680	1.0117	.9082	.5867	1.1880
.7	.2927	1.0194	.9091	.6515	1.2157
.8	.3149	1.0321	.9146	.7212	1.2483
.9	.3351	1.0531	.9289	.8043	1.2912
1.0	.3536	1.1390	1.0000	1.0000	1.4142

A diagram, given in Fig. 8, shows that as κ increases from 1 to ∞ the trapezoid changes continuously from a square to an equilateral triangle. The two following theorems are evident:

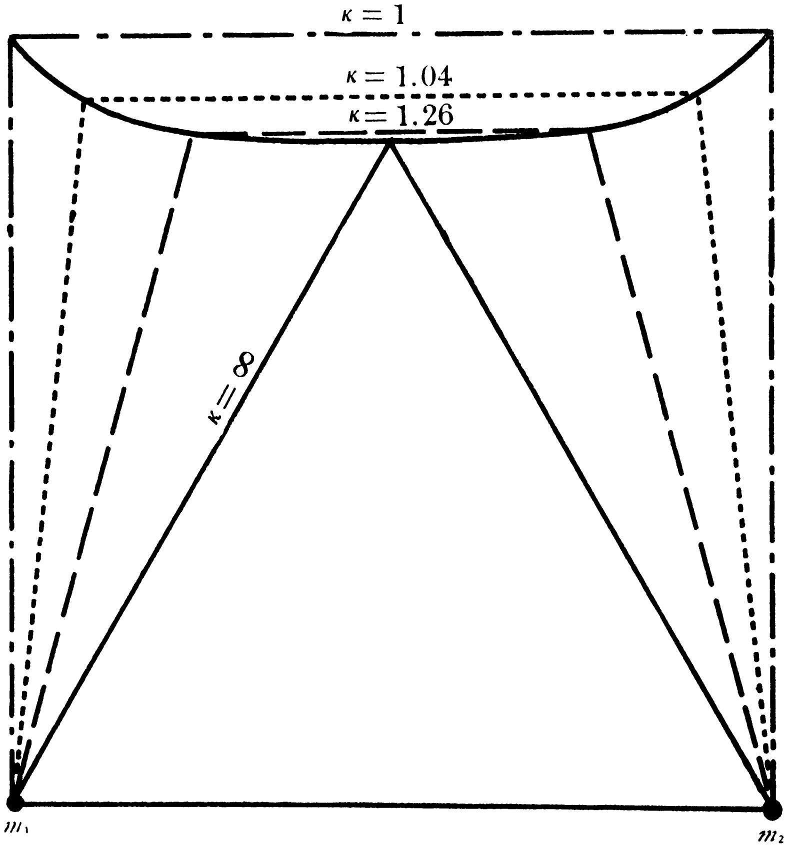

Fig. 8

THEOREM. *There exists one and only one isosceles trapezoid for given values of r_0 and r_1.*

THEOREM. *There exists one and only one isoceles trapezoid for a given value of an interior angle, provided this angle lies between* 60° *and* 120°.

The mass ratios. From equations (17) it is found that

$$\frac{m_2}{m_1} = -\frac{\alpha_1}{\beta_1}\frac{S_4}{S_6}, \qquad \frac{m_4}{m_3} = -\frac{\alpha_3}{\beta_3}\frac{S_2}{S_6}.$$

For the trapezoid, however,

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_3}{\beta_3} = +1,$$

and, since $R_2 = R_4$ and $R_5 = R_6$, equations (19) show that

$$S_2 = S_4 = -S_6;$$

hence

$$m_1 = m_2, \text{ and } m_3 = m_4.$$

Also, from (17) and (7.5),

$$\frac{m_3}{m_1} = -\frac{\beta_3}{\beta_1}\frac{S_1}{S_2} = -\frac{\alpha_2}{\alpha_4}\frac{S_1}{S_2}.$$

For the isosceles trapezoid $-\alpha_4 = \beta_2$; hence

$$\frac{m_4}{m_2} = \frac{m_3}{m_1} = \frac{\alpha_2}{\beta_2}\frac{S_1}{S_2} = \frac{r_1}{r_3}\frac{S_1}{S_2} = -\frac{r_1}{r_3}\frac{S_1}{S_6},$$

which by virtue of (33) becomes

$$\frac{m_3}{m_1} = \frac{r_1}{r_3}\left(\frac{S_1}{S_3}\right)^{1/2} = \left(\frac{r_3}{r_1}\cdot\frac{r_0^3 - r_1^3}{r_0^3 - r_3^3}\right)^{1/2}.$$

For $r_3 = 0$ this ratio vanishes. As r_3 increases, both fractions of the radicand increase and have the limit unity. Hence as the trapezoid changes from the equilateral triangle to the square, the ratio m_3/m_1 increases steadily from zero to one. Hence

THEOREM. *For every $m_1 = m_2 > 0$ and $m_3 = m_4 > 0$, there exists one and only one isosceles trapezoid configuration.*

19. Quadrilaterals in the neighborhood of isosceles trapezoids. On returning to Fig. 4 with the assumption that $r_3 \leqq r_0$, it is seen that in any given diagram, that is, $r_1 \leqq r_0$ given, there exists one and only one isosceles trapezoid point in each of the regions Oa_1b_1 and Oa_2b_2, and each of these points is the reflection of the other in a plane P which passes through O and is perpendicular to r_1. For other admissible quadrilaterals m_4 lies in or on the boundary of Oa_1b_1 and m_3 within or on the boundary of Oa_2b_2.

Suppose m_4 approaches the boundary Oa_1 and that $r_1 < r_0$; then R_6 approaches R_0, and on account of the dynamical equations

$$(41)\quad (R_1 - R_0)(R_3 - R_0) = (R_2 - R_0)(R_4 - R_0) = (R_5 - R_0)(R_6 - R_0),$$

R_3 also tends toward R_0 and so also does R_4, provided m_4 is not approaching either of the points O and a_1. That is, if m_1 approaches the arc Oa_1 (end points excepted), the point m_3 approaches the arc Ob_2, and if m_4 is on Oa_1, the point m_3 is on the arc Ob_2, and $r_2=r_3=r_5=r_0$. The points m_2, m_3, and m_4 form an equilateral triangle and the mass m_1 vanishes. Now let m_4 move along the arc a_1O toward the point O. The point m_3 simultaneously moves along the arc Ob_2 toward the point b_2. If the point m_4 is at the point O, the point m_3 may be anywhere on the arc b_2a_2, which, extended, passes through m_1. Thus m_1, m_2, and m_3 lie on the arc of a circle whose center is at m_4. The corresponding three masses are all zero unless m_3 is at the end points a_2 or b_2.

Now let the point m_4 move along the arc Ob_1. The point m_3 simultaneously moves along the arc a_2O. The triangle $m_1m_4m_3$ is equilateral, and the mass m_2 is zero, and so on. It is seen that as m_4 moves around the boundary of its region counterclockwise, m_3 also moves around the boundary of its region, but in a clockwise direction; and throughout all of the motion r_3 is constant and equal to r_0.

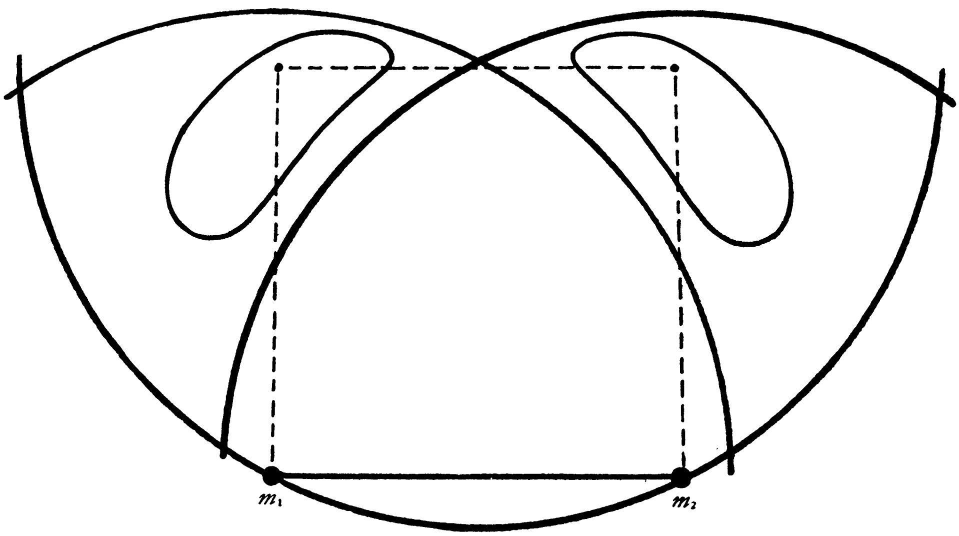

Fig. 9

If m_4 and m_3 are permitted to move in their respective regions but subject to the condition that $r_3<r_0$ is constant, it is found that each point traces a closed curve in its own region and that each of these curves is the reflection of the other in the plane P, Fig. 9. Thus there is a family of curves $r_3=C<r_0$. These curves shrink in length as the constant C diminishes; in fact, they shrink down upon the isosceles trapezoid points as a limit (for the proof see §21). Consequently, the smallest possible value of r_3 for given values of r_0 and r_1 is that one which belongs to the isosceles trapezoid.

20. **Conjugate curves.** If the point m_3 of an admissible quadrilateral describes a curve in the region S_3, m_1 and m_2 remaining fixed, the point m_4 of the admissible quadrilateral also describes a curve in the region S_4. Suppose these curves are reflected in a plane which passes through O, Fig. 4, and is perpendicular to r_1; then if m_3 moves along the reflected curve in S_3, m_4 will move along the reflected curve in S_4. These curves are conjugate curves. Do there exist conjugate curves that are unaltered by reflection? The answer is in the affirmative since the curves $r_3=$const. obviously are of this type. Such a pair of curves are self conjugate. Let C_3 and C_4 be the members of a self conjugate pair; that is, C_3 is the reflection of C_4, and conversely. If m_3 lies on C_3 at a certain point p, m_4 lies on C_4 at a certain point π. The reflection of π lies on C_3 and will be denoted by $\bar{p}$; then p and $\bar{p}$ are conjugate points on C_3.

Since

$$(R_2 - R_0)(R_4 - R_0) = (R_0 - R_5)(R_0 - R_6) \tag{41.5}$$

for every admissible quadrilateral, and since for conjugate points $R_4=\overline{R}_2$ and $R_6=\overline{R}_5$, it follows that on every self conjugate curve

$$(R_2 - R_0)(\overline{R}_2 - R_0) = (R_0 - R_5)(R_0 - \overline{R}_5).$$

Since

$$\bar{r}_5 = r_6, \quad \bar{r}_2 = r_4, \quad \text{and} \quad \bar{r}_4 = r_2,$$

for the reflected quadrilaterals, equations (8) and (9) give

$$\alpha_1\bar{r}_5^2 = -\alpha_1\beta_1 r_1^2 + \beta_3^2 r_3^2 + \beta_1\bar{r}_2^2,$$

$$\beta_1 r_5^2 = +\alpha_1\beta_1 r_1^2 - \alpha_3^2 r_3^2 + \alpha_1 r_2^2,$$

and the sum of these gives the relation

$$\alpha_1\bar{r}_5^2 + \beta_1 r_5^2 = -(\alpha_3+\beta_3)r_2^2 + (\beta_1\bar{r}_2^2 + \alpha_1 r_2^2), \tag{42}$$

which holds at the conjugate points p and $\bar{p}$. If p and $\bar{p}$ tend toward coincidence, r_3 tends toward parallelism with r_1, and in the limit α_1, β_1, α_3 and β_3 are infinite; but

$$\lim\,(\alpha_3/\alpha_1) = \beta_3/\alpha_1 = r_1/r_3, \qquad \lim\,(\beta_1/\alpha_1) = +1,$$

so that, when the two points coincide, equation (42) becomes

$$r_5^2 = r_1 r_3 + r_2^2,$$

and equation (41) becomes

$$(R_2 - R_0)^2 = (R_0 - R_5)^2.$$

Since these are the equations which define the isosceles trapezoid points, it follows that *every self conjugate curve, on which the conjugate points have a point of coincidence, passes through an isosceles trapezoid point.*

As an example of a self conjugate curve of this last type consider the locus of positions of m_3 for which

$$R_2 - R_0 = R_0 - R_5.$$

From the dynamical equation (41.5) it follows that the conjugate curve is defined by the relation

$$(R_4 - R_0) = (R_0 - R_6),$$

which is the reflection of the locus for m_3. Hence these curves are self conjugate and they all pass through the isosceles trapezoid points. The curves for which $r_3 = \text{const.}$ are self conjugate, but they do not pass through the isosceles trapezoid points. They are closed curves which contain the isosceles trapezoid points in their interiors.

21. **Power series solutions in the neighborhood of isosceles trapezoids.** In order to investigate the properties of the self conjugate curves $r_3 = \text{const.}$ near an isosceles trapezoid point, a power series expansion of the solutions of the geometrical and dynamical equations will be useful.

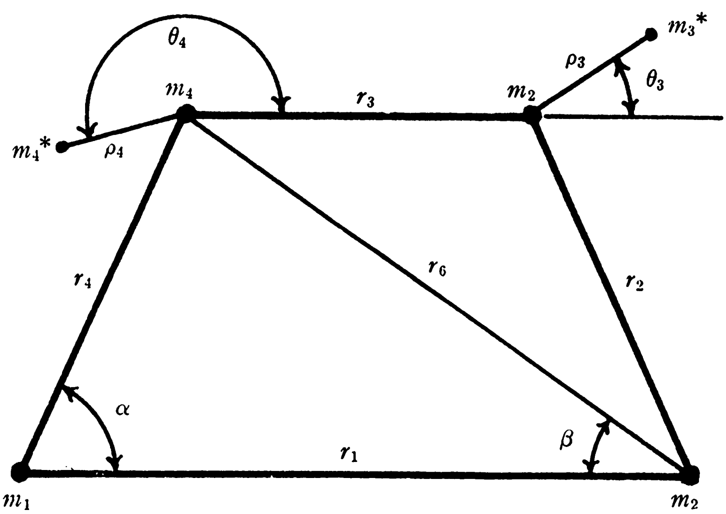

Fig. 10

Consider two points m_3^* and m_4^* of a solution which is near the isosceles trapezoid points m_3 and m_4 which are associated with the trapezoid r_1, $r_2 = r_4$, r_3, $r_5 = r_6$. Let the point m_3^* be defined by the polar coördinates ρ_3, θ_3 with origin at m_3, the polar axis being the line m_4 to m_3. Likewise let the point m_4^* be defined by the polar coördinates ρ_4, θ_4 with the same polar axis but with origin at m_4, Fig. 10. Let r_1, r_2^*, r_3^*, r_4^*, r_5^*, and r_6^* be the sides and diagonals of the quadrilateral associated with corners m_1, m_2, m_3^*, and m_4^*. Let α be the angle between r_1 and r_4, and β the angle between r_1 and r_6. Then

$$
\begin{aligned}
r_3^{*2} &= (r_3 + \rho_3 \cos \theta_3 - \rho_4 \cos \theta_4)^2 + (\rho_3 \sin \theta_3 - \rho_4 \sin \theta_4)^2, \\
r_2^{*2} &= r_2^2 + \rho_3^2 - 2r_2\rho_3 \cos (\theta_3 + \alpha), \\
r_4^{*2} &= r_2^2 + \rho_4^2 + 2r_2\rho_4 \cos (\theta_4 - \alpha), \\
r_5^{*2} &= r_5^2 + \rho_3^2 + 2r_5\rho_3 \cos (\theta_3 - \beta), \\
r_6^{*2} &= r_5^2 + \rho_4^2 - 2r_5\rho_4 \cos (\theta_4 + \beta).
\end{aligned} \tag{43}
$$

Since the point m_4 is uniquely defined if m_3 is given, it follows that ρ_4 vanishes if ρ_3 vanishes. If these values are substituted in the dynamical equations (41), bearing in mind, of course, that

$$R_i^* = (r_i^*)^{-3},$$

and the equations are then expanded in powers of ρ_3 and ρ_4 there result the two equations

$$
\begin{aligned}
&\frac{-3(R_1 - R_0)R_3}{r_3}(\rho_3 \cos \theta_3 - \rho_4 \cos \theta_4) + \cdots \\
&= \frac{3(R_2 - R_0)R_2}{r_2}(\rho_3 \cos (\theta_3 + \alpha) - \rho_4 \cos (\theta_4 - \alpha)) + \cdots \\
&= \frac{3(R_0 - R_5)R_5}{r_5}(\rho_3 \cos (\theta_3 - \beta) - \rho_4 \cos (\theta_4 + \beta)) + \cdots,
\end{aligned} \tag{44}
$$

the terms independent of ρ_3 and ρ_4 vanishing by themselves, since they belong to the isosceles trapezoid solution. The solution of the linear terms of these equations gives as a first approximation to the complete solution

$$\rho_4 = \rho_3, \qquad \theta_4 = -\theta_3.$$

For further development let two new variables κ and λ be defined as follows:

$$
\begin{aligned}
\kappa &= \frac{\rho_4 - \rho_3}{\rho_3} = \kappa_1\rho + \kappa_2\rho^2 + \cdots, \\
\lambda &= \theta_3 + \theta_4 = \lambda_1\rho + \lambda_2\rho^2 + \cdots, \\
\rho_3 &= \rho, \qquad \theta_3 = \theta,
\end{aligned} \tag{45}
$$

since κ and λ are evidently expansible as a convergent power series in ρ, the coefficients being functions of θ. With these variables, ρ, θ, κ, λ, the expansion of the first equation of (43), up to and including terms of the second degree in ρ, is

$$r_3^{*2} = r_3^2 + 2r_3(-\rho\kappa\cos\theta + \rho\lambda\sin\theta) + 4\rho^2\sin^2\theta + \cdots; \tag{46}$$

and similar expansions can be obtained for the remaining sides and diagonals. When these expansions are substituted into the dynamical equations (41), not only the terms independent of ρ are satisfied by themselves, but the linear terms also. Since κ and λ occur only in terms of at least the second degree in ρ, κ, and λ, each of which contains ρ as a factor, the factor ρ can be divided out, and there remain the two equations

$$\begin{aligned}
&\frac{(R_1-R_0)R_3}{r_3}\left[\kappa\cos\theta - \lambda\sin\theta - \frac{2\rho\sin^2\theta}{r_3} + \cdots\right] \\
&= \frac{(R_2-R_0)R_2}{r_2}\left[-\kappa\cos(\theta+\alpha) + \lambda\sin(\theta+\alpha) - \frac{\rho(1-5\cos^2(\theta+\alpha))}{r_2}\right] \\
&\quad - 3\frac{R_2^2}{r_2^2}\rho\cos^2(\theta+\alpha) + \cdots \\
&= \frac{(R_0-R_5)R_5}{r_5}\left[-\kappa\cos(\theta-\beta) - \lambda\sin(\theta-\beta) + \frac{\rho(1-5\cos^2(\theta-\beta))}{r_5}\right] \\
&\quad - 3\frac{R_5^2}{r_5^2}\rho\cos^2(\theta-\beta) + \cdots.
\end{aligned} \tag{47}$$

These two equations can be solved for κ and λ uniquely as a power series in ρ, as indicated in (45), provided the determinant of the coefficients of κ and λ is different from zero. This determinant is

$$\begin{aligned}
&\frac{(R_2-R_0)(R_0-R_5)R_2R_5}{r_2r_5}\sin(\alpha+\beta) + \frac{(R_1-R_0)(R_0-R_5)R_3R_5}{r_3r_5}\sin\beta \\
&\quad + \frac{(R_1-R_0)(R_2-R_0)R_3R_2}{r_3r_2}\sin\alpha.
\end{aligned}$$

In §16 it was shown that $60° < \alpha < 120°$ and $\beta < 60°$. Consequently $\alpha+\beta < 180°$, and all of the terms of the determinant are positive. It cannot, therefore, vanish. The coefficients in the expansions of equations (45) can be computed, but, for the sake of brevity, the computations will be omitted. They are found to be periodic functions of θ with the period 2π, a result that could have been anticipated from the geometrical relations between the conjugate points. If these expansions are substituted in (46), and the coefficients simplified by simple trigonometric relations between the sides and angles of the trapezoid, it is found that

$$r_3^* = r_3 + 0 \cdot \rho$$

$$(48)\quad + \frac{r_3(5R_0 - 2R_2)\cos^2(\theta+\alpha) + r_3(5R_0 - 2R_5)\cos^2(\theta-\beta) + 2r_1(R_2 - R_0)\sin^2\theta}{R_3(R_1 - R_0)(R_2^2 + R_5^2) + r_1 r_3(R_2 - R_0)}\rho^2$$

$$+ \cdots .$$

From the relations between the sides and diagonals of admissible quadrilaterals, §14, and equation (34), it follows that all of the coefficients in equation (48) are positive. Therefore for values of ρ different from zero r_3^* is greater than r_3. This establishes analytically the limit property of the isosceles trapezoid points discussed in §19.

The locus of m_3^*, for $r_3^* = \text{const.}$, up to terms of the second degree in ρ, is

$$\rho^2[r_3(5R_0 - 2R_2)\cos^2(\theta+\alpha) + r_3(5R_0 - 2R_5)\cos^2(\theta-\beta) + 2r_1(R_2 - R_0)\sin^2\theta]$$
$$+ \cdots = \text{a constant.}$$

The expression within the bracket is a homogeneous quadratic function of $\sin\theta$ and $\cos\theta$ that does not vanish. Therefore for ρ sufficiently small the self conjugate curves $r_3 = \text{const.}$ are approximately ellipses with centers at the isosceles trapezoid points. The conjugate ellipse described by m_4^* is, of course, the reflection of that described by m_3^*, but if m_3^* moves in its ellipse clockwise, m_4^* moves in its ellipse counterclockwise, so that the position of m_4^* in its ellipse is not the reflection of m_3^* in its ellipse.

22. **Masses associated with admissible quadrilaterals.** It was shown in §§14 and 17 that the ratios of the masses which are associated with a given admissible quadrilateral and a given r_0 are uniquely determined by equations (26). Indeed, if the six sides of the quadrilateral are given, r_0 itself is uniquely determined by equation (18), unless all three of the expressions for R_0 take an indeterminate form, and this can happen only if

$$R_1 = R_2 = R_5 \text{ and } R_2 = R_4 = R_6,$$

or

$$r_1 = r_2 = r_5 \text{ and } r_2 = r_4 = r_6.$$

In this case r_0 is entirely arbitrary; the masses m_1, m_2, and m_3 are equal and lie at the vertices of an equilateral triangle; the mass m_4 is arbitrary but it is placed at the center of the equilateral triangle which is formed by the other three masses.

THEOREM. *Associated with each admissible quadrilateral there is one and only one set of mass ratios, with the single exception of three equal masses at the vertices of an equilateral triangle and a fourth arbitrary mass at the center of gravity of the other three.*

With one exception, if an admissible quadrilateral is given the mass ratios are uniquely determined. If the masses are given, does there necessarily exist an admissible quadrilateral for them, and, if so, is this quadrilateral unique?

Consider first convex quadrilaterals for which the condition

$$r_0 > r_1 > 0$$

holds, and suppose r_0 is given. For every position of m_3 in the region Oa_2b_2, Fig. 4, there exists one and only one position for m_4, and that position lies in the region Ob_1a_1. The mass ratios $m_1:m_2:m_3:m_4$ can be regarded as functions of the position of m_3, for both the mass ratios and the ratios α_i/β_i, $i=1, 2, 3, 4$, are continuous single-valued functions of the position of m_3. The ratios

$$\frac{\alpha_i}{\beta_i} = 0, \text{ or } \infty \qquad (i = 1, 2, 3, 4)$$

implies that at least three of the bodies are in a straight line. Hence it is evident from Fig. 4 that for every $r_1 \neq 0$ that is less than r_0 the ratios α_i/β_i have a positive finite upper bound and a positive non-zero lower bound. If m_3 approaches any point on the arc Ob_2 (the point O itself excepted), the ratio $m_1:m_2$ approaches $0:1$, by equation (27). If m_3 approaches any point of the arc Oa_2 (the point O again excepted), the ratio $m_1:m_2$ approaches $0:1$. Therefore there exists a curve C_3 which starts at O and terminates on the arc a_2b_2 on which the ratio $m_1:m_2=k_{12}$ is an arbitrarily given positive constant.

As m_3 moves along the curve C_3, the point m_4 moves along a certain curve C_4 in the region Ob_1a_1. If m_3 approaches O on C_3 the point m_4 approaches a definite point on the arc a_1b_1, and the ratio $m_3:m_4$ approaches $1:0$. If m_3 approaches the arc a_2b_2 along the curve C_3, the ratio $m_3:m_4$ approaches $0:1$. Hence there exists a point on the curve C_3 at which $m_3:m_4=k_{34}$, an arbitrarily given constant. Hence

THEOREM. *For every r_0 and r_1 such that $r_0>r_1>0$, there exists at least one admissible quadrilateral for which*

$$\frac{m_1}{m_2} = k_{12} \textit{ and } \frac{m_3}{m_4} = k_{34},$$

where k_{12} and k_{34} are arbitrarily given positive constants.

From the symmetry of the figure, it follows that *for every $r_3<r_0$ there exists an admissible quadrilateral for which*

$$\frac{m_1}{m_2} = k_{12} \textit{ and } \frac{m_3}{m_4} = k_{34}.$$

Now consider the series of admissible convex quadrilaterals for which

$$\frac{m_1}{m_2} = k_{12} \text{ and } \frac{m_3}{m_4} = k_{34},$$

as, r_0 remaining fixed, r_1 tends toward zero. It follows immediately from the geometry of the quadrilateral, Fig. 1, that as r_1 tends toward zero, r_5 tends toward r_2, and r_6 tends toward r_4. But in every admissible convex quadrilateral

$$r_2 < r_0 < r_5 \text{ and } r_4 < r_0 < r_6.$$

Consequently, at the limit

$$r_2 = r_4 = r_5 = r_6 = r_0;$$

and since

$$(R_1 - R_0)(R_3 - R_0) = (R_2 - R_0)(R_4 - R_0),$$

the left member tends toward zero just as the right member does. But R_1 tends toward infinity; hence R_3 tends toward R_0, which is the same as saying that r_3 tends toward r_0. Consequently as r_1 tends toward zero, all of the remaining sides and the diagonals tend toward the value r_0.

The functions α_2 and β_4, however, tend toward zero as r_1 diminishes; for the point A in Fig. 1 tends toward coincidence with the point m_2. But since, §5,

$$r_1\mathbf{a}_1 + \alpha_2 r_2 \mathbf{a}_2 = \beta_4 r_4 \mathbf{a}_4,$$

it is seen that

$$\mathbf{a}_1 + \frac{\alpha_2}{r_1} r_2 \mathbf{a}_2 = \frac{\beta_2}{r_1} r_4 \mathbf{a}_4,$$

and consequently as r_1 tends toward zero, the ratios α_2/r_1 and β_4/r_1 tend toward limits that are not zero. For the angles between the unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, and between $\mathbf{a}_4$ and $\mathbf{a}_1$ for admissible convex quadrilaterals, always lie between 60° and 120°, and the angle between $\mathbf{a}_4$ and $\mathbf{a}_2$ tends toward the definite limit 120°. The common limit of r_2 and r_4 is r_0, and the coefficient of $\mathbf{a}_1$ in the above equation is unity. Hence zero is not a limit for either α_2/r_1 or β_4/r_1.

Now, by equations (26),

$$\frac{m_1}{m_3} = \frac{\alpha_4\beta_2}{\beta_4\alpha_2}\,\frac{R_3 - R_0}{R_1 - R_0}\,\frac{m_4}{m_2} = \frac{\alpha_4\beta_2}{\left(\dfrac{\beta_4\alpha_2}{r_1^2}\right)}\,\frac{r_1(R_3 - R_0)}{1 - r_1^3 R_0}\,\frac{m_4}{m_2}.$$

Since the limit of α_2 and β_4 is zero, the limits of β_2 and α_4 are -1 and $+1$ respectively. Hence, as r_1 tends toward zero, r_0 remaining fixed, the limit of

$$\frac{m_1 m_2}{m_3 m_4} = 0.$$

In a similar manner it is shown that if r_1 tends toward r_0, then r_3 tends toward zero, and the limit of

$$\frac{m_3 m_4}{m_1 m_2} = 0.$$

It follows, therefore, that for some value of r_1 in the interval

$$0 \leqq r_1 \leqq r_0$$

there exists an admissible convex quadrilateral for which

$$\frac{m_1}{m_2} = k_{12}, \qquad \frac{m_3}{m_4} = k_{34}, \text{ and } \frac{m_1 m_2}{m_3 m_4} = k,$$

where k_{12}, k_{34}, and k are arbitrarily specified constants. It will be observed that it is not proved that there exists but one such value of r_1.

This result can be expressed as follows:

THEOREM. *For every four given masses and assigned order there exists at least one admissible convex quadrilateral.*

A corresponding theorem for concave quadrilaterals has not been proved.

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